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Asymmetric Inequality for Non-homogeneous Ternary Quadratic Forms

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Let $Q(x, y, z)$ be an indefinite ternary quadratic form of determinant $D < 0$. Let $t \geq 0$ be any given real number. Then the author proves the existence of a function $f(t)$ such that given any reals x_0, y_0, z_0 we can find integers x, y, z such that $-t(f(t)|D|)^{1/3} < Q(x+x_0, y+y_0, z+z_0) \leq (f(t)|D|)^{1/3}$. The result is best possible for eight values of t and in particular includes the previous best known results as special cases.

1. INTRODUCTION

Let $Q(x_1, \ldots, x_n)$ be an indefinite quadratic form in n-variables with signature $(r, n-r)$, $0 < r < n$ and determinant $D \neq 0$. The symmetric non-homogeneous problem is to find the best possible constant $C_{r,n-r}$ such that given any real numbers $c_1, \ldots, c_n$ we can find integers $x_1, \ldots, x_n$ such that

$$|Q(x_1+c_1, \ldots, x_n+c_n)| \leq (C_{r,n-r}|D|)^{1/n} \tag{1.1}$$

The value $C_{1,1} = 1/4$ follows from the classical result of Minkowski on product of the non-homogeneous linear forms. $C_{2,1} = C_{1,2} = 27/100$ are due to H. Davenport [6]. The value $C_{r,r} = 1/4$ for all r is due to B. J. Birch [3]. The result $C_{3,1} = C_{1,3} = 1/3$ were proved by the author [9]. G. L. Watson [13] proved the result for all $n \geq 21$ and any r, $0 < r < n$.

One could ask the more general question: Is there a function $f(t)$ for all t in $0 \leq t < \infty$ or for some special t such that given any real numbers $c_1, \ldots, c_n$ we can find integers $x_1, \ldots, x_n$ such that

$$-t(f(t)|D|)^{1/n} < Q(x_1+c_1, \ldots, x_n+c_n) \leq (f(t)|D|)^{1/n} \tag{1.2}$$

One would naturally like to find best possible $f(t)$. In practice even the

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$f(t)$ which work for all t will be best possible only for some special values of t. For forms of the type (1, 1) results of this type were proved by several authors (Davenport and Heilbronn [7], Blaney [4], Barnes and Swinnerton-Dyer [2]). For forms of the type (2, 1), $f(0)$, $f(1)$ and $f(\infty)$ were obtained by Barnes [1], Davenport [6] and the author [8] respectively. The author [9], [10] has also obtained $f(0)$ when Q is of the type (3, 1) and 2, 2).

Our object is to obtain a function $f(t)$ for all t in $(0 \le t \le \infty)$ (the case $t = \infty$ has to be viewed as a limiting case as $t \to \infty$) for ternary forms of the type (2, 1); which is best possible for eight values of t and in particular includes the previous known results (i.e. $f(0)$, $f(1)$ and $f(\infty)$). More precisely we prove:

THEOREM. *Let $t \ge 0$ and $Q(x, y, z)$ be an indefinite ternary quadratic form of the type* (2, 1) *and determinant $D < 0$. Then given any real numbers x_0, y_0, z_0 we can find $(x, y, z) \equiv (x_0, y_0, z_0)$ (mod 1) such that*

$$-t(f(t)|D|)^{1/3} < Q(x, y, z) \le (f(t)|D|)^{1/3} \tag{1.3}$$

where $f(t)$ is given by

$$\begin{aligned}
&\frac{4}{(1+t)^2(1+5t)} && \text{for } 0 \le t \le \frac{1}{7},\\
&\frac{32}{(1+t)(7-t)(1+9t)} && \text{for } \frac{1}{7} \le t \le \gamma = \frac{151-26\sqrt{6}}{115},\\
&\frac{27}{(1+4t)(11-t)(1+t)} && \text{for } \gamma \le t \le 1,\\
&\frac{27}{t(t+9)^2} && \text{for } 1 \le t \le \frac{61}{35},\\
&\frac{1}{(1+t)^2} && \text{for } \frac{61}{35} < t \le 7\\
&\frac{8}{(1+t)^3} && \text{for } t \ge 7.
\end{aligned}$$

Equality occurs if and only if

$t = 0$;	$Q \sim \rho Q_1$	$= \rho(x^2+yz)$;	$(x_0, y_0, z_0) \equiv (0, 0, 0)$ (mod 1)
or	$Q \sim \rho Q_2$	$= \rho(x^2+y^2+2z^2)$;	$(x_0, y_0, z_0) \equiv (1/2, 1/2, 1/2)$ (mod 1)
$t = 1/7$;	$Q \sim \rho Q_3$	$= \rho(x^2+y^2-3z^2)$;	$(x_0, y_0, z_0) \equiv (1/2, 1/2, 1/2)$ (mod 1)
$t = 3/5$;	$Q \sim \rho Q_4$	$= \rho(x^2+4yz)$;	$(x_0, y_0, z_0) \equiv (1/2, 1/2, 1/2)$ (mod 1)
$t = 1$;	$Q \sim \rho Q_5$	$= \rho(x^2+10y^2+xz-z^2)$;	$(x_0, y_0, z_0) \equiv (0, 1/2, 0)$ (mod 1)
$t = 9/7$;	$Q \sim \rho Q_6$	$= \rho(x^2+9y^2-3z^2)$;	$(x_0, y_0, z_0) \equiv (1/2, 1/2, 1/2)$ (mod 1)
$t = 3$;	$Q \sim \rho Q_7$	$= \rho(x^2+yz)$;	$(x_0, y_0, z_0) \equiv (1/2, 1/2, 1/2)$ (mod 1)

or $\quad Q \sim \rho Q_8 = \rho(2x^2+y^2-z^2); \quad (x_0, y_0, z_0) \equiv (1/2, 1/2, 1/2) \pmod 1$

$t = 7; \quad Q \sim \rho Q_9 = \rho(x^2+y^2-z^2); \quad (x_0, y_0, z_0) \equiv (1/2, 1/2, 1/2) \pmod 1$

or $\quad Q \sim \rho Q_{10} = \rho(x^2+2yz); \quad (x_0, y_0, z_0) \equiv (1/2, 0, 0) \pmod 1$

$t = \infty; \quad Q \sim \rho Q_{11} = \rho(x^2+xy+y^2-2yz); \quad (x_0, y_0, z_0) \equiv (0, 0, 1/2) \pmod 1$

where $\rho > 0$.

By simple congruence considerations it is easy to verify that equality is needed for the forms Q_i at the appropriate points.

2. Some Basic Lemmas

Lemma 2.1. *Let $Q(x, y, z)$ be an indefinite ternary quadratic form of determinant $D < 0$. Then there exist integers u, v, w such that*

$$0 < Q(u, v, w) \leq (9/4|D|)^{1/3} \tag{2.1}$$

except when $Q(x, y, z) \sim \rho(x^2+yz)$, $\rho > 0$.

This is Theorem 1 of Oppenheim [12].

Lemma 2.2. *Let $\phi(y, z)$ be an indefinite binary quadratic form with discriminant $\Delta^2 > 0$ and let $\lambda > 0$ be a real number. Then given any real numbers y_0, z_0, there exist $(y, z) \equiv (y_0, z_0) \pmod 1$ such that*

$$-\frac{\Delta}{4\lambda} \leq \phi(y, z) < \frac{\lambda\Delta}{4} \tag{2.2}$$

with equality if and only if $\lambda^2 = m/(m+2)$, $m = 1, 2, 3, \ldots$ $\phi(y, z) \sim c\phi_m(y, z) = c[m(m+2)y^2-z^2]$; and $(y_0, z_0) \equiv (m/2, 1/2)$ $\pmod 1$; $c > 0$.

This is Lemma 3 of Davenport [6] and Theorem 1 of Blaney [4].

Lemma 2.3. *Let $\phi(y, z)$ be an indefinite binary quadratic form with discriminant $\Delta^2 > 0$. Let $0 \leq \mu \leq 1/3$ be a real number. Then for any real numbers y_0, z_0 we can find $(y, z) \equiv (y_0, z_0) \pmod 1$ such that*

$$-\frac{\mu\Delta}{[(1+\mu)(1+9\mu)]^{1/2}} \leq \phi(y, z) < \frac{\Delta}{[(1+\mu)(1+9\mu)]^{1/2}} \tag{2.3}$$

Equality occurs if and only if $\mu = 1/(4n-1)$, $n = 1, 2, \ldots$ and
$\phi \sim c\phi_n = c(ny^2-(n+2)z^2)$; $(y_0, z_0) \equiv (1/2, 1/2) \pmod 1$; or
$\mu = 0$ and $\phi \sim c\phi' = cyz$; $(y_0, z_0) \equiv (0, 0) \pmod 1$ or
$\phi \sim c\phi'' = c(y^2-z^2)$; $(y_0, z_0) \equiv (1/2, 1/2) \pmod 1$; where $c > 0$.

This is Theorem 2 of Blaney [4].

LEMMA 2.4. *Let* $0 \leq \nu < 1$ *be a given real number;* $\phi(y, z)$ *an indefinite binary quadratic form of discriminant* $\Delta^2 > 0$. *Then for any real numbers* y_0, z_0 *there exist* $(y, z) \equiv (y_0, z_0) \pmod 1$ *satisfying*

$$\frac{\nu^2\Delta}{[(1-\nu)^3(1+3\nu)]^{1/2}} \leq \phi(y, z) < \frac{\Delta}{[(1-\nu)^3(1+3\nu)]^{1/2}} \tag{2.4}$$

This is Theorem 3 of Blaney [4].

LEMMA 2.5. *Let* α, β, d *be real numbers with* $d \geq 1$. *Then given any real number* x_0 *there exists* $x \equiv x_0 \pmod 1$ *such that*

$$0 < (x+\alpha)^2 - \beta^2 \leq d \tag{2.5}$$

provided

$$\beta^2 \begin{cases} \leq \left(\dfrac{d-1}{2}\right)^2 & \text{if } d \text{ is an integer} \\ < \left(\dfrac{[d]}{2}\right)^2 & \text{if } d \text{ is not an integer.} \end{cases} \tag{2.6}$$

Strict inequality in (2.6) *implies strict inequality in* (2.5).

This is Lemma 6 of author [10].

LEMMA 2.6. *Let* α, β, d *be real numbers with* $\beta^2 > 1/4$ *and* $d \geq 1$. *Then for any real number* x_0 *we can find* $x \equiv x_0 \pmod 1$ *satisfying*

$$0 < -(x+d)^2 + \beta^2 \leq d \tag{2.7}$$

provided that

$$\beta^2 \begin{cases} \leq \left(\dfrac{d+1}{2}\right)^2 & \text{if } d \text{ is an integer} \\ < \left(\dfrac{[d]}{2}\right)^2 + d & \text{if } d \text{ is not an integer.} \end{cases} \tag{2.8}$$

Strict inequality in (2.8) *implies strict inequality in* (2.7).

This is Lemma 2 of author [8].

3. PROOF OF THE THEOREM

Let

$$m = \inf_{\substack{u, v, w \text{ integers.} \\ Q(u,v,w) > 0}} Q(u, v, w) \tag{3.1}$$

Then by Lemma 1, $0 \leq m < (9/4|D|)^{1/3}$ except when $Q \sim m(x^2+yz)$.

LEMMA 3.1. *If $m = 0$ and μ is any real number, then for any given real numbers x_0, y_0, z_0 there exist $(x, y, z) \equiv (x_0, y_0, z_0) \pmod 1$ such that $|Q(x, y, z)-\mu|$ is arbitrarily small.*

Proof. The result is in fact true for all indefinite quadratic form in three or more variables. For a proof see B. J. Birch [3].

Thus our theorem is true when $m = 0$.

LEMMA 3.2. If $Q \sim m(x^2+yz)$, *then the theorem is true.*

Proof. Without loss of generality we can suppose $Q(x, y, z) = x^2+yz$. Let

$$d = (f(t)|D|)^{1/3} = \left(\frac{f(t)}{4}\right)^{1/3} \tag{3.2}$$

It can be easily verified that

(i) $(1+t)d > 1$ if $t > 3$.

(ii) Since $f(t)$ is decreasing we have $f(t) \geq f(3) = 1/16$ for $0 \leq t \leq 3$, i.e. $d \geq 1/4$ with equality only if $t = 3$.

(iii) $(1+t)d > 1/2$ for $0 \leq t \leq 3$.

If $(1+t)d > 1$, choose $x \equiv x_0 \pmod 1$ arbitrarily $y \equiv y_0 \pmod 1$ with $0 < y \leq 1$ and then choose $z \equiv z_0 \pmod 1$ to satisfy

$$0 < x^2+yz+td \leq y \leq 1 < (1+t)d$$

or

$$-td < x^2+yz < d.$$

If $(1+t)d \leq 1$, then $0 \leq t \leq 3$. If $(y_0, z_0) \not\equiv (0, 0) \pmod 1$, without loss of generality suppose $y_0 \not\equiv 0 \pmod 1$. Choose $y \equiv y_0 \pmod 1$ such that $0 < |y| \leq 1/2$. Choose $x \equiv x_0 \pmod 1$ arbitrarily and then choose $z = z_0 \pmod 1$ to satisfy

$$0 < x^2+yz+td \leq |y| \leq 1/2 < (1+t)d$$

so that (1.3) is satisfied.

Let now $(y_0, z_0) \equiv (0, 0) \pmod 1$. If $0 < t \leq 3$, take $y = z = 0$ and $x \equiv x_0 \pmod 1$ with $|x| \leq 1/2$, so that

$$-td < 0 \leq x^2+yz = x^2 \leq 1/4 \leq d,$$

with equality only if $t = 3$ and $x_0 \equiv 1/2 \pmod 1$.

If $t = 0$, take $y = z = 0$ and $x \equiv x_0 \pmod 1$ with $0 < x \leq 1$, so that $0 < x^2+yz = x^2 \leq 1 = d$ with equality only if $x_0 \equiv 0 \pmod 1$. This completes the proof of the theorem if $Q \sim m(x^2+yz)$.

We may now suppose $m > 0$ and $Q(x, y, z) \nsim \rho(x^2+yz)$. Given $0 < \varepsilon_0 < 1/16$, there exist integers u, v, w such that

$$Q(u, v, w) = \frac{m}{1-\varepsilon}, \quad 0 \leq \varepsilon < \varepsilon_0 < 1/16.$$

By Lemma 2.1 we can further suppose that

$$Q(u,v,w) = \frac{m}{1-\varepsilon} \le (9/4|D|)^{1/3} \tag{3.3}$$

Also by definition of m since $\varepsilon_0 < 1/16$ we must have $(u, v, w) = 1$. By applying a suitable unimodular substitution we can suppose $Q(1,0,0) = m/(1-\varepsilon)$. Then we can write $Q(x, y, z)$ as

$$Q(x,y,z) = \frac{m}{1-\varepsilon}[(x+hy+gy)^2+\phi(y,z)]$$

where $\phi(y, z)$ is an indefinite binary quadratic form with discriminant

$$\Delta^2 = \frac{4|D|}{\left(\frac{m}{1-\varepsilon}\right)^3} \ge \frac{16}{9}$$

Also there is no loss of generality in supposing

$$|h| \le 1/2, \quad |g| \le 1/2.$$

By definition of m we have

$$(x+hy+gz)^2+\phi(y,z) \ge 1-\varepsilon \text{ or } \le 0$$

for all integers x, y, z. Because of homogeneity it suffices to prove:

THEOREM A. *Let $\varepsilon > 0$ be sufficiently small. Let $Q(x,y,z) = (x+hy+gz)^2 + \phi(y,z)$ where $\phi(y,z)$ is an indefinite binary quadratic form of discriminant*

$$\Delta^2 = 4|D| \ge 16/9 \tag{3.4}$$

and

$$|h| \le 1/2, \quad |g| \le 1/2; \tag{3.5}$$

such that for integers x, y, z we have either

$$Q(x,y,z) \ge 1-\varepsilon \quad or \quad \le 0 \tag{3.6}$$

Let

$$d = (f(t)|D|)^{1/3} \tag{3.7}$$

Then given any real numbers x_0, y_0, z_0 we can find

$$(x,y,z) \equiv (x_0, y_0, z_0) \pmod 1$$

such that

$$-td < Q(x,y,z) \le d \tag{3.8}$$

Equality occurs if and only if $Q \sim Q_i$ stated in the theorem.

Remarks. It can be easily verified that $4d-1 > 0$ for $t \le 3$. We shall use this fact repeatedly.

4. Proof of Theorem A

LEMMA 4.1. *If $Q(x, y, z)$ satisfies the conditions of Theorem* A *then for integers y, z we have either*

$$\phi(y,z)=0 \quad or \quad \phi(y,z)\geq \tfrac{3}{4}-\varepsilon \quad or \quad \phi(y,z)\leq -\tfrac{1}{4} \tag{4.1}$$

Proof. From (3.6) it follows easily that $\phi(y, z)$ cannot lie in the intervals $(0, \frac{3}{4}-\varepsilon)$ and $(-\frac{1}{4}, -\varepsilon)$. If $-\varepsilon \leq \phi(yz) < 0$ choose smallest integer n such that

$$\phi(ny, nz) = n^2\phi(y, z) < -\varepsilon.$$

Then

$$n > 1,\ n^2\phi(y, z) < -\varepsilon \leq (n-1)^2\phi(y, z),$$

so that

$$-\varepsilon > \phi(ny, nz) = n^2\phi(y, z) > \left(\frac{n}{n-1}\right)^2(-\varepsilon) > -\tfrac{1}{4},$$

since $\varepsilon < 1/16$ and $n^2/(n-1)^2 \leq 4$ for all n. But this is impossible for integers ny, nz. Hence the lemma must be true.

LEMMA B. *Let $Q(x, y, z)$ satisfy the conditions of Theorem* A. *Suppose we can find $(y, z) \equiv (y_0, z_0)$ (mod 1) such that*

$$-v_2 \leq \phi(y, z) < v_1 \tag{4.2}$$

where

$$v_1 = d-\tfrac{1}{4} \tag{4.3}$$

$$v_2\begin{cases} < \left(\dfrac{[(1+t)d]}{2}\right)^2 + td & \text{if } (1+t)d \text{ is not an integer} \\ = \left(\dfrac{(1+t)d-1}{2}\right)^2 + td & \text{if } (1+t)d \text{ is an integer.}\end{cases} \tag{4.4}$$

Then we can find $x \equiv x_0$ (mod 1) such that

$$-td < Q(x, y, z) \leq d \tag{4.5}$$

Further strict inequality in (4.2) *implies strict inequality in* (4.5).

Proof. Let $\alpha = hy+gz$, so that $Q(x, y, z) = (x+\alpha)^2+\phi(y, z)$. If $(1+t)d < 1$, choose $x \equiv x_0$ (mod 1) with $|x+\alpha| \leq \frac{1}{2}$, so that

$$-td < v_2 \leq Q(x, y, z) < \tfrac{1}{4}+v_1 = d.$$

Let now $(1+t)d \geq 1$. If $-td < \phi(y, z) < v_1$ proceed as above. Therefore suppose

$$0 \leq \beta^2 = -td-\phi(y, z) \leq v_2-td$$

so that

$$Q(x, y, z)+td = (x+\alpha)^2-\beta^2$$

Then the lemma follows from Lemma 2.5 with d replaced by $(1+t)d$.

Thus in order to prove Theorem A, it suffices to satisfy the condition (4.2) of Lemma B; and to show that equality can occur only for the forms equivalent to Q_i. In all cases we shall not give details of the second part which is easy to verify.

LEMMA 4.2. *If* $(1+t)d \leq 1$, *then* $5/19 \leq t \leq 8/7$ *and in that case we have*

$$\frac{d^3}{f(t)(4d-1)} < td \tag{4.6}$$

Then $(y, z) \equiv (y_0, z_0) \pmod 1$ *can be chosen to satisfy* (4.2) *with strict inequality.*

Proof. Using the fact that $d = f(t)|D|)^{1/3} \geq (4/9f(t))^{1/3}$ and the various values of $f(t)$ it is easy to verify that $(1+t)d > 1$ if $t \leq 1/7$ and $t > 61/35$. If $1 \leq t \leq 61/35$ *then* $(1+t)d > 1$ if $27(1+t)^3|D|/t(t+9)^2 > 1$. Since $|D| \geq 4/9$, this is so if

$$g(t) = 11t^3+18t^2-45t+12 > 0.$$

Since $g(0) > 0$, $g(1) < 0$, $g(8/7) > 0$, by Descarte's rule of signs $g(t) > 0$ for $t \geq 8/7$. Thus $(1+t)d \leq 1$ implies $1 \leq t \leq 8/7$. In that case (4.6) holds if

$$f(d) = d^2-4tf(t)d+tf(t) < 0 \tag{4.7}$$

for $k = (4/9f(t)^{1/3} \leq d \leq 1/(1+t)$. A slight computation shows that

$$f(0) > 0, f(k) < 0, f(1/(1+t)) < 0, f(\infty) > 0 \quad \text{for} \quad 1 \leq t \leq 8/7.$$

Consequently (4.7) holds. This proves (4.6) for $t \geq 1$. If $1/7 \leq t \leq \gamma$, then

$$(1+t)^3d^3 = \frac{32(1+t)^2}{(7-t)(1+9t)}|D| \geq \frac{128(1+t)^2}{9(7-t)(1+9t)} > 1$$

if

$$209t^2-302t+65 = (19t-5)(11t-13) > 0 \quad \text{or} \quad t < 5/19.$$

Thus if $(1+t)d \leq 1$ then $t \geq 5/19$. In this case (4.6) holds if

$$g(d) = \frac{d^2}{4d-1} < tf(t).$$

It can be easily verified that $g(d)$ is increasing for $t < \gamma$, so that for $(1+t)d \leq 1$, we have

$$g(d) \leq g\left(\frac{1}{1+t}\right) = \frac{1}{(1+t)(3-t)} < tf(t) = \frac{32t}{(1+t)(7-t)(1+9t)}$$

if $h(t) = 23t^2-34t-7 < 0$, which can be easily verified to be true for $5/19 \leq t < \gamma$ by the rule of signs.

If $\gamma \leq t \leq 1$, then (4.6) holds for $(1+t)d \leq 1$ if

$$g(d) = d^2 - 4tf(t)d + tf(t) < 0 \tag{4.8}$$

for $k = (4/9f(t))^{1/3} \leq d \leq 1/(1+t)$.

It can be easily verified by a simple computation that

$$g(0) > 0,\ g(k) < 0,\ g\left(\frac{1}{1+t}\right) < 0,\ g(\infty) > 0,$$

so that by the rule of signs (4.8) holds.

Thus we have shown that if $(1+t)d \leq 1$, then $5/19 \leq t \leq 8/7$ and then (4.6) holds. By Lemma 2.2 with $\lambda = (4d-1)/\Delta > 0$ (it can be easily verified that $d > 1/4$ in $5/19 \leq t \leq 8/7$) we can find $(y, z) \equiv y_0, z_0)$ (mod 1) such that

$$-td < -\frac{d^3}{f(t)(4d-1)} = -\frac{\Delta}{4\lambda} \leq \phi(y, z) < \frac{\lambda\Delta}{4} = d - \tfrac{1}{4}.$$

Thus (4.2) is satisfied and the result follows from Lemma B.

Thus we can now suppose $(1+t)d > 1$.

LEMMA 4.3. *If* $0 \leq t \leq 1$ *and* $2 < (1+t)d$ *then the result holds with*

$$f(t) = \frac{27}{(1+t)(4t+1)(11-t)}.$$

Proof. It may be easily seen that $\lambda = (4d-1)/\Delta > 0$ and by Lemma (2.2) we can find $(y, z) \equiv (y_0, z_0)$ (mod 1) so that

$$-\frac{d^3}{f(t)(4d-1)} \leq \phi(y, z) < d - \tfrac{1}{4}. \tag{4.9}$$

The result will follow from Lemma B if we have

$$\frac{d^3}{f(t)(4d-1)} < \begin{cases} \left(\dfrac{(1+t)d-1}{2}\right)^2 + td & \text{if } (1+t)d \geq 3 \\ 1 + td & \text{if } 2 < (1+t)d < 3 \end{cases} \tag{4.10}$$

Case (i). $(1+t)d \geq 3$.

One can easily see that (4.9) holds if

$$g(d) = \frac{((3-t)d-1)^2}{4d^3} < (1+t)^2 - \frac{1}{f(t)}$$

Simple calculations on $g'(d)$ show that $g(d)$ is decreasing for $(1+t)d \geq 3$ if $0 \leq t \leq 1$. Thus

$$g(d) \leq g\left(\frac{3}{1+t}\right) = \frac{4}{27}(2-t)^2(1+t) = (1+t)^2 - \frac{1}{f(t)}$$

and the result holds in this case.

Case (ii). $2 < (1+t)d < 3$.
In this case (4.10) holds if

$$g(d) = \frac{(1+td)(4d-1)}{d^3} > \frac{1}{f(t)}$$

It is easy to show that $g'(d) < 0$ for $2 < (1+t)d < 3$ and $0 \leq t \leq 1$. Thus

$$g(d) > g\left(\frac{3}{t+1}\right) = \frac{(1+t)(4t+1)(11-t)}{27} = \frac{1}{f(t)}.$$

This completes the proof of the lemma.

LEMMA 4.4. *If* $0 \leq t \leq 1/7$ *then the theorem is true.*

Proof. For $0 \leq t \leq 1/7$,

$$f(t) = \frac{4}{(1+t)^2(1+5t)} > \frac{27}{(1+t)(1+4t)(11-t)},$$

hence for $(1+t)d > 2$ the lemma follows from Lemma 4.3.
Let now $1 < (1+t)d \leq 2$. Let

$$\mu = \frac{-5+\left\{16+\dfrac{9.64d^3}{f(t)(4d-1)^2}\right\}^{1/2}}{9}$$

so that

$$\frac{\Delta}{\{(1+\mu)(1+9\mu)\}^{1/2}} = d-\tfrac{1}{4}.$$

It can be easily shown that $0 \leq \mu \leq 1/3$ for $0 \leq t \leq 1/7$. Therefore by Lemma 2.3 there exist $(y, z) \equiv (y_0, z_0) \pmod 1$ such that

$$-\mu(d-\tfrac{1}{4}) \leq \phi(y, z) < d-\tfrac{1}{4}.$$

The result will follow from Lemma B if we have

$$\mu(d-\tfrac{1}{4}) \leq \tfrac{1}{4}+td \tag{4.11}$$

and the equality corresponds to the listed critical forms.
It is easy to see that (4.11) holds if

$$g(d) = \frac{2+(1+9t)d}{4d^2} \geq \frac{1}{f(t)(1+t)}.$$

Clearly for $1 < (1+t)d \leq 2$ we have

$$g(d) \geq g\left(\frac{2}{1+t}\right) = \frac{(1+5t)(1+t)}{4} = \frac{1}{f(t)(1+t)}$$

and the lemma follows, since it is not difficult to show that equality occurs only for the forms listed in the theorem.

LEMMA 4.5. *Theorem is true for* $1/7 \leq t \leq 1$.

Proof. As before for $(1+t)d > 2$ the result follows from Lemma 4.3 since $f(t) \geq 27/(1+t)(1+4t)(11-t)$ in $1/7 \leq t \leq 1$. Let now $1 < (1+t)d \leq 2$. By applying Lemma 2.2 with $\lambda = (4d-1)/\Delta > 0$, we can find $(y, z) \equiv (y_0, z_0) \pmod 1$ such that

$$-\frac{d^3}{f(t)(4d-1)} \leq \phi(y, z) < d - \tfrac{1}{4}.$$

The result will follow from Lemma B if we have

$$g(d) = \frac{(1+4td)(4d-1)}{4d^3} \geq \frac{1}{f(t)}. \tag{4.12}$$

Since $g(d)$ is decreasing for $1 < (1+t)d \leq 2$ (4.12) is satisfied because

$$g(d) \geq g\left(\frac{2}{1+t}\right) = \frac{(1+9t)(7-t)(1+t)}{32} \geq \frac{1}{f(t)}$$

as can be easily verified. One can easily show that equality occurs only in the cases listed in the theorem. This proves Lemma 4.5.

LEMMA 4.6. *Theorem is true for* $1 \leq t \leq 61/35$.

Proof. By applying Lemma 2.2 with $\lambda = (4d-1)/\Delta > 0$, we can find $(y, z) \equiv (y_0, z_0) \pmod 1$ such that

$$-\frac{d^3}{f(t)(4d-1)} \leq \phi(y, z) < d - \tfrac{1}{4}.$$

The result will follow from Lemma B if we have

$$\frac{d^3}{f(t)(4d-1)} \begin{cases} < \left(\dfrac{(1+t)d-1}{2}\right)^2 + td & \text{for } (1+t)d > 2 \\ \leq \tfrac{1}{4} + td & \text{for } 1 < (1+t)d \leq 2 \end{cases} \tag{4.13}$$

with equality only for critical forms.

For $(1+t)d > 2$, (4.13) is satisfied if

$$g(d) = \frac{((3-t)d-1)^2}{4d^3} \leq (1+t)^2 - \frac{1}{f(t)}.$$

It is easy to verify that for $(1+t)d > 2$

$$g(d) \leq \max\left\{g\left(\frac{2}{1+t}\right), g\left(\frac{3}{3-t}\right)\right\} = g\left(\frac{3}{3-t}\right) \text{ for } 1 \leq t \leq \frac{61}{35}$$

$$= \frac{(3-t)^3}{27} = (1-t)^2 - \frac{1}{f(t)}$$

and the result follows in this case.

For $1 < (1+t)d \leq 2$, (4.13) is satisfied if

$$\frac{1}{f(t)} \leq \frac{(1+4td)(4d-1)}{4d^3}$$

or

$$g(d) = \frac{4}{f(t)}d^3 - 16td^2 + 4(t-1)d + 1 \leq 0$$

which is easily seen to be true for $1 < t \leq 61/35$. Thus the lemma is proved.

LEMMA 4.7. *Theorem is true for* $61/35 < t \leq 7$.

Proof.

Case (i): $d > 1/4$. Let $c = (1+t)d$. Let

$$\frac{1}{\mu} = -5 + \left[16 + \frac{64d^3}{f(t)(4d-1)}\right]^{1/2}, \tag{4.14}$$

so that

$$\frac{\mu\Delta}{[(1+\mu)(1+9\mu)]^{1/2}} = d - \tfrac{1}{4}.$$

It can be easily verified that $\mu \leq 1/3$ for $t \geq 5/3$. By applying Lemma 2.3 to $-\phi(y, z)$ we can find $(y, z) \equiv (y_0, z_0) \pmod 1$ such that

$$-(d-\tfrac{1}{4}) < -\phi(y, z) \leq \frac{1}{\mu}(d-\tfrac{1}{4})$$

or

$$-\frac{1}{\mu}(d-\tfrac{1}{4}) \leq \phi(y, z) < d - \tfrac{1}{4}.$$

The required result then follows from Lemma B if we have

$$\frac{1}{\mu}(d-\tfrac{1}{4}) \begin{cases} \leq \left(\dfrac{(1+t)d-1}{2}\right)^2 + td & \text{if } (1+t)d = c \text{ is an integer} \\ < \left(\dfrac{[(1+t)d]}{2}\right)^2 + td & \text{if } c \text{ is not an integer.} \end{cases} \tag{4.15}$$

For $c > 3$, (4.15) is satisfied if we have

$$\frac{1}{\mu}(d-\tfrac{1}{4}) \leq \left(\frac{(1+t)d-1}{2}\right)^2 + td.$$

A slight simplification shows that this is so if

$$(1+t)c^2 - 2(13-3t)c + 8(1+t) \geq 0 \quad \text{for} \quad c > 3$$

which is true if $t \geq 61/35$.

If $2 < c \leq 3$, (4.15) is true if

$$\frac{1}{\mu}(d-\tfrac{1}{4}) \leq 1+td, \text{ or}$$

$$f(c) = 64c^3 - 16(t-9)c^2 - 8(11-t)c + 15(1+t) \leq 0$$

Since $f(-\infty) < 0$, $f(0) > 0$, $f(2) < 0$, $f(3) = 3(-35t+61) < 0$, $f(\infty) > 0$, by the rule of signs we have $f(c) < 0$ for $2 < c \leq 3$.

If $1 < c \leq 2$, then the result holds if

$$\frac{1}{\mu}(d-\tfrac{1}{4}) < \tfrac{1}{4}+td,$$

or

$$(c-2)(4c-(1+t)) \leq 0,$$

which is so since $c \leq 2$ and $c = (1+t)d > (1+t)/4$ for $d > 1/4$. It is not difficult to show that equality can occur only for the critical forms. This completes the proof of the lemma in this case.

Case (ii). $d \leq 1/4$, $(1+t)d > 1$.

Then $t > 3$ and $(1+t)d \leq 2$ since $t \leq 7$.

Let $0 \leq v < 1$ be the root of

$$\frac{v^2\Delta}{\{(1-v)^3(1+3v)\}^{1/2}} = \tfrac{1}{4}-d.$$

Then by Lemma 2.5 we can find $(y, z) \equiv (y_0, z_0) \pmod 1$ such that

$$\frac{v^2\Delta}{\{(1-v)^3(1+3v)\}^{1/2}} < -\phi(y, z) \leq \frac{\Delta}{\{(1-v)^3(1+3v)\}^{1/2}}$$

or

$$-\frac{1}{v^2}(\tfrac{1}{4}-d) \leq \phi(y, z) < d-\tfrac{1}{4}.$$

To prove the result it suffices to show that

$$\frac{1}{v^2}(\tfrac{1}{4}-d) \leq \tfrac{1}{4}+td, \quad \text{since} \quad 1 < (1+t)d \leq 2$$

or

$$v^2 \geq \frac{1-4d}{1+4td} = v_0^2, \quad \text{say}. \tag{4.16}$$

Since

$$g(v) = \frac{(1-v)^3(1+3v)}{v^4} = \frac{64d^3(1+t)^2}{(1-4d)^2}$$

and $g(v)$ is decreasing in $0 \leq v < 1$, it suffices to show that

$$g(v_0) \geq \frac{64d^3(1+t)^2}{(1-4d)^2}$$

or

$$\frac{(1-v_0)^3(1+3v_0)}{v_0^4} > \frac{64d^3(1+t)^2}{(1-4d)^2} = \frac{(1-v_0^2)^3(1+t)^2}{(1+tv_0)^3\left(1-\dfrac{1-v_0^2}{1+tv_0^2}\right)^2} = \frac{(1-v_0^2)^3}{(1+tv_0^2)v_0^4}$$

or $(3t-1)v_0 > 3-t$.

This is clearly true since $t > 3$ and $v_0 \geq 0$. It is easy to show that equality can occur only for the special forms Q_i.

5. Proof of Theorem A continued: $t > 7, f(t) = 8/(1+t)^3$

LEMMA 5.1. *If $d > 1/4$, then the theorem is true.*

Proof. Let $c = (1+t)d$. Let

$$\lambda = -5+\left(16+\frac{8c^3}{(4d-1)^2}\right)^{1/2} \tag{5.1}$$

be the positive root of

$$(\lambda+1)(\lambda+9) = \frac{16\Delta^2}{(4d-1)^2} = \frac{8c^3}{(4d-1)^2}.$$

Then it can be easily verified that $\lambda \geq 3$ for $t > 7$.

By applying Lemma 3.3 with $\mu = 1/\lambda$ to $-\phi(y, z)$ we can find $(y, z) \equiv (y_0, z_0) \pmod 1$ such that

$$-\frac{\Delta}{[(1+\lambda)(9+\lambda)]^{1/2}} < -\phi(y, z) \leq \frac{\lambda\Delta}{[(1+\lambda)(9+\lambda)]^{1/2}}$$

so that

$$-\lambda(d-\tfrac{1}{4}) \leq \phi(y, z) < d-\tfrac{1}{4}.$$

The result will follow from Lemma B if we have

$$\lambda(d-\tfrac{1}{4}) < \left(\frac{(1+t)d-1}{2}\right)^2 + td.$$

A slight simplification shows that this is satisfied if

$$g(c) = c^3 + \frac{4(7-t)}{1+t}c^2 + \frac{4(15-t)}{1+t}c - 16 > 0. \tag{5.2}$$

It can be easily verified that $g(c)$ is increasing, so that

$$g(c) = g((1+t)d) \geq g\left(\frac{1+t}{4}\right) = \frac{(1+t)}{64}(t-7)^2 > 0 \quad \text{for} \quad t > 7.$$

Hence (5.2) is satisfied and the Lemma is proved.

We can now suppose $d < 1/4$.

It will be convenient to write $\phi(y, z)$ as $-\phi_1(y, z)$ since the rest of the proof is quite similar to the method of proof in [8].

It suffices to prove that we can find $(y, z) \equiv (y_0, z_0) \pmod 1$ such that

$$\tfrac{1}{4}-d < \phi_1(y, z)\begin{cases} \leq \left(\dfrac{c-1}{2}\right)^2 + td & \text{if } c \text{ is an integer} \\ < \left(\dfrac{[c]}{2}\right)^2 + td & \text{if } c \text{ is not an integer} \end{cases} \tag{5.3}$$

and that the equality occurs only in the special cases.

By Markoff Chain theorem (see e.g. [5], chapter 2) we can find integers u, v such that

$$|a = \phi_1(u, v)| \leq \Delta/\sqrt{5}, \quad (u, v) = 1. \tag{5.4}$$

Further if $|a| > \Delta/3$, then $\phi_1(y, z)$ is a Markoff form and so represents both a and $-a$. In that case we shall assume $a > 0$. By Lemma 4.1 we have the following three possibilities:

(i) $0 < a \leq \Delta/\sqrt{5}$

(ii) $-1/3\Delta \leq a \leq -\frac{3}{4}+\varepsilon$

(iii) $a = 0$.

If $0 < a \leq \Delta/\sqrt{5}$, by replacing ϕ_1 by an equivalent form we can suppose

$$\phi_1(y, z) = a(y+fz)^2 - \left(\frac{\Delta^2}{4a}\right)z^2.$$

Note: $c = (1+t)d \geq (32/9)^{1/3}$ and $\Delta^2 = c^3/2$.

Choose $z \equiv z_0 \pmod 1$ such that $|z| \leq 1/2$. The result for case $0 < a \leq \Delta/\sqrt{5}$ follows from

LEMMA 5.2. *We can choose* $y \equiv y_0 \pmod 1$ *so that* (5.3) *holds with strict inequality if*

$$\text{(i) } 0 < a < \gamma_1 = \left(\frac{c+1}{2}\right)^2 + \tfrac{1}{4} - 2d - \left[\frac{c^3}{8} + \frac{(1-4d)(c+1)^2-4d}{4}\right]^{1/2} \tag{5.5}$$

$$\textit{or } \text{(ii) } \gamma_1 \leq a \leq \Delta/\sqrt{5} \quad \textit{and} \quad c > 2. \tag{5.6}$$

$$\textit{or } \text{(iii) } 0 < a < \gamma_2 = c + \frac{1-4d}{2} - \left\{\frac{c^3}{8} + c(1-4d) + \left(\frac{1-4d}{2}\right)^2\right\}^{1/2} \tag{5.7}$$

and $(32/9)^{1/3} \leq c \leq 2$

$$\textit{or } \text{(iv) } \gamma_2 \leq a \leq \Delta/\sqrt{5} \quad \textit{and} \quad (32/9)^{1/3} \leq c \leq 2. \tag{5.8}$$

Proof of 5.2 (i). If (5.5) holds, choose $y \equiv y_0 \pmod 1$ to satisfy

$$\left(\frac{1-4d}{4a}+\frac{\Delta^2}{4a^2}z^2\right)^{1/2} < y+fz < \left(\frac{1-4d}{4a}+\frac{\Delta^2}{4a^2}z^2\right)^{1/2}+1 \tag{5.9}$$

so that

$$\tfrac{1}{4}-d < \phi_1(y,z) \le \tfrac{1}{4}-d+a+\left\{a(1-4d)+\frac{c^3}{8}\right\}^{1/2}. \tag{5.10}$$

Thus the result holds if we have

$$\tfrac{1}{4}-d+a+\left[a(1-4d)+\frac{c^3}{8}\right]^{1/2} < \left(\frac{c-1}{2}\right)^2+td.$$

A slight computation shows that this is so if (5.5) is true.
Hence the lemma is proved in this case.

Proof of 5.2 (ii). In this case we have

$$\begin{aligned}\frac{1-4d}{4a}+\frac{\Delta^2}{4a^2}z^2 &\le \frac{1-4d}{4a}+\frac{\Delta^2}{16a^2}\\ &< \frac{1-4d}{4\gamma_1}+\frac{c^3}{2(4\gamma_1)^2}\\ &= \left[\frac{\left(\frac{c^3}{2}+(1-4d)\left((c+1)^2-\frac{4c}{1+t}\right)\right)^{1/2}-(1-4d)}{4\gamma_1}\right]^2 < 1.\end{aligned}$$

A slight computation shows that this is so if

$$g(c) = 2c^3-c^2+\frac{24}{1+t}c^2+2c+\frac{48}{1+t}c-12 > 0.$$

This is easily seen to be true for $c \ge 2$. Thus we have

$$\frac{1-4d}{4a}+\frac{\Delta^2}{4a^2}z^2 < 1.$$

If

$$\frac{1-4d}{4a}+\frac{\Delta^2}{4a^2}z^2 < \tfrac{1}{4},$$

choose

$$y \equiv y_0 \pmod 1 \quad \text{with} \quad \tfrac{1}{2} \le |(y+fz)| \le 1$$

and if

$$\tfrac{1}{4} \le \frac{1-4d}{4a}+\frac{\Delta^2}{4a^2}z^2 < 1,$$

choose

$$y \equiv y_0 \pmod 1 \quad \text{with} \quad 1 \le |y+fz| \le 3/2,$$

so that in either case we will have

$$\tfrac{1}{4}-d < a(y+fz)^2-\frac{\Delta^2}{4a}z^2 = \phi_1(y, z) \le 2a+\tfrac{1}{4}-d$$
$$\le \frac{2\Delta}{\sqrt{5}}+\tfrac{1}{4}-d$$
$$< \left(\frac{c-1}{2}\right)^2+td$$

if $8\Delta/\sqrt{5} < c(c+2)$ or $c^2-\frac{12}{5}c+4 = (c-\frac{6}{5})^2+\frac{64}{25} > 0$, which is so.

Hence by Lemma B the result follows.

Proof of 5.2 (iii). In this case $(32/9)^{1/3} \le c \le 2$. Choose $y \equiv y_0 \pmod 1$ as in proof of 5.2 (i) so that (5.10) holds. Since $[c] = 1$, (4.2) holds if we have

$$\tfrac{1}{4}-d+a+\left[a(1-4d)+\frac{c^3}{8}\right]^{1/2} < \tfrac{1}{4}+td.$$

A slight computation shows that this is true if (5.7) holds. Hence the result holds in this case.

Proof of 5.2 (iv). For $\gamma_2 \le a \le \Delta/\sqrt{5}$, $(32/9)^{1/3} \le c \le 2$, it can again be easily verified that

$$\frac{1-4d}{4a}+\frac{\Delta^2}{4a^2}z^2 < \frac{1}{4\gamma_2}+\frac{\Delta^2}{16\gamma_2^2} < 1$$

and the result follows as in 5.2 (ii).

This completes the proof of Lemma 5.2.

Lemma 5.3. *If $a = -b$, $\frac{9}{4}-\varepsilon \le b \le \frac{1}{3}\Delta$, then* (5.3) *can be satisfied with strict inequality.*

Proof. Without loss of generality we can suppose

$$\phi_1(y, z) = -b(y+fz)^2+\frac{\Delta^2}{4b}z^2.$$

Also $c^3 = 2\Delta^2 \ge 18b^2 \ge 2(9/4-3\varepsilon)^2 > 10$ if ε is taken sufficiently small. We shall consider two subcases:

(i) $\frac{9}{4}-\varepsilon \le b \le \Delta/3$, $c^3 > 288/25$

(ii) $\frac{9}{4}-\varepsilon \le b \le \Delta/3$, $10 < c^3 \le 288/25$.

Proof in case (i). In this case it is easy to verify that

$$b \le \frac{\Delta}{3} < \frac{(\Delta^2+1)^{1/2}-1}{2}. \tag{5.11}$$

Choose $z \equiv z_0 \pmod 1$ with $1/2 \leq |z| \leq 1$, so that

$$\frac{\Delta^2}{4b^2}z^2 - \frac{1-4d}{4b} \geq \frac{\Delta^2}{16b^2} - \frac{1}{4b} > \tfrac{1}{4},$$

by using (5.11).

If

$$\tfrac{1}{4} < \frac{\Delta^2}{4b^2}z^2 - \frac{1-4d}{4b} \leq 1,$$

choose $y \equiv y_0 \pmod 1$ with $|y+fz| \leq 1/2$ so that

$$\tfrac{1}{4} - d < \phi_1(y, z) \leq b + \tfrac{1}{4} - d \leq (\Delta/3) + \tfrac{1}{4} - d < \left(\frac{c-1}{2}\right)^2 + td$$

if $\Delta/3 < c(c+2)/4$;
or $c^2 + \frac{28}{9}c + 4 > 0$, which is clearly so. Hence the result holds in this case.

If

$$\frac{\Delta^2}{4b^2}z^2 - \frac{1-4d}{4b} > 1,$$

choose $y \equiv y_0 \pmod 1$ with

$$\left(\frac{\Delta^2}{4b^2}z^2 - \frac{1-4d}{4b}\right)^{1/2} - 1 \leq y + fz < \left(\frac{\Delta^2}{4b^2}z^2 - \frac{1-4d}{4b}\right)^{1/2},$$

so that

$$\begin{aligned}\tfrac{1}{4} - d < \phi(y, z) &\leq (\Delta^2 z^2 - (1-4d)b)^{1/2} - b + \tfrac{1}{4} - d \\ &< (\Delta^2 - (\tfrac{3}{4} - \varepsilon)(1-4d))^{1/2} - \tfrac{3}{4} + \varepsilon + \tfrac{1}{4} - d \\ &< \Delta - \tfrac{1}{2} - d + \varepsilon.\end{aligned}$$

Thus the required inequality is satisfied if we have

$$\Delta - \tfrac{1}{2} - d < \left(\frac{c-1}{2}\right)^2 + td$$

or $f(c) = c^2(c-2)^2 + 6c^2 + 12c + 9 > 0$, which is true for all c. Hence the result follows from Lemma B.

Proof in case (ii). $10 < c^3 \leq 288/25$, $[c] = 2$.

Choose $z \equiv z_0 \pmod 1$ with $1 \leq |z| \leq 3/2$, so that

$$\frac{\Delta^2}{4b^2}z^2 - \frac{1-4d}{4b} \geq \frac{\Delta^2}{4b^2} - \frac{1}{4b} > \tfrac{1}{4},$$

since $b \leq \Delta/3$, $\Delta > 1$.

If

$$\tfrac{1}{4} < \frac{\Delta^2}{4b^2}z^2 - \frac{1-4d}{4b} \leq 1,$$

the result follows as above. Let now

$$\frac{\Delta^2}{4b^2} z^2 - \frac{1-4d}{4b} > 1.$$

Since $2 < c < 3$, we have the desired result if we can find $y \equiv y_0 \pmod 1$ such that

$$\tfrac{1}{4} - d < -b(y+fz)^2 + \frac{\Delta^2}{4b} z^2 < 1 + td$$

or

$$0 < -(y+fz)^2 + \left(\frac{\Delta^2}{4b^2} z^2 - \frac{1-4d}{4b}\right) < \frac{3+4c}{4b}. \tag{5.12}$$

It is easy to verify that in this case we have $(3+4c)/4b > 3$, so that by Lemma 2.6 (5.12) can be satisfied if we have

$$\frac{\Delta^2}{4b^2} z^2 - \frac{1-4d}{4b} < \left(\frac{3}{2}\right)^2 + \frac{3+4c}{4b}.$$

Since $|z| \leq 3/2$, this will be so if we have

$$\frac{9\Delta^2}{16b^2} < \frac{9}{4} + \frac{3+4c}{4b}$$

or

$$9b^2 + (3+4c)b \geq \frac{9c^3}{8}. \tag{5.13}$$

Since $b > \frac{3}{4} - \varepsilon$, we have

$$9b^2 + (3+4c)b \geq \frac{81}{16} + \frac{9}{4} + 3c + O(\varepsilon) > \frac{9c^3}{8}$$

can be easily verified for $10 < c^3 \leq 288/25$ and sufficiently small ε. Thus (5.13) is satisfied and the lemma is proved.

LEMMA 5.4. *If $a = 0$, then again the theorem is true.*

Proof. By a suitable substitution we can suppose that

$$\phi_1(y, z) = \lambda y(z - \theta y), \quad \lambda > 0. \tag{5.14}$$

Then to prove the result we have to find $(y, z) \equiv (y_0, z_0) \pmod 1$ such that

$$-(d - \tfrac{1}{4}) < \phi(y,z) < \left(\frac{c-1}{2}\right)^2 + td$$

or

$$0 < \lambda y(z - \theta y) + d - \tfrac{1}{4} \leq \frac{c(c+2)}{4}. \tag{5.15}$$

Choose $y \equiv y_0 \pmod 1$ such that $0 < y \leq 1$. Now choose $z \equiv z_0 \pmod 1$ such that

$$0 < \lambda y(z-\theta y)+d-\tfrac{1}{4} \leq \lambda y \leq \lambda$$

and (5.15) follows if

$$4\lambda \leq c(c+2)$$

or

$$8c^3 \leq c^2(c+2)^2$$

or

$$(c-2)^2 \geq 0,$$

which is true for all c. By considering the cases of equality more closely it is easy to verify that equality occurs only for the special forms stated in the theorem. This completes the proof of Theorem A.

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